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Orientifolds, unoriented instantons and localization

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Abstract: We consider world-sheet instanton effects in $\mathcal{N} = 1$ string orientifolds of non-compact toric Calabi-Yau threefolds. We show that unoriented closed string topological amplitudes can be exactly computed using localization techniques for holomorphic maps with involution. Our results are in precise agreement with mirror symmetry and large N duality predictions.

Keywords: Superstring Vacua, Differential and Algebraic Geometry.

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Contents

1. Introduction

It is widely known that oriented closed string instantons play a central role in the $\mathcal{N}=2$ dynamics of Calabi-Yau compactifications. In particular one can obtain exact results for the prepotential and more general couplings by counting (in an appropriate sense) holomorphic maps from the world-sheet to the target space.

Similar results have been obtained in the past years for $\mathcal{N}=1$ D-brane vacua, in which case the nonperturbative effects are due to open string instantons [1]–[14]. Although this case is more intricate from a mathematical point of view, open string enumerative techniques have been developed up to the point of concrete computations. From a physical point of view these effects often generate a nonperturbative superpotential, with the effect of removing the vacuum degeneracy of the models in question.

Another class of $\mathcal{N}=1$ string vacua which has been less studied so far consists of orientifolds of Calabi-Yau threefolds equipped with an antiholomorphic involution [15]–[20]. Even in the absence of D-branes, these models exhibit interesting nonperturbative effects due to closed string unoriented instantons. So far exact results for topological unoriented string amplitudes on the resolved conifold have been predicted in [19] as a result of large N duality. Moreover, the $\mathbb{RP}^2$ result has been checked using local mirror symmetry in [15].

The purpose of the present paper is to develop a systematic approach to this problem based on an unoriented version of closed string enumerative geometry. More precisely we propose a method of summing unoriented world-sheet instantons based on localization of holomorphic maps with involution. Our approach is similar to the open string **A**-model techniques of [6, 7, 11] which rely on localization with respect to a circle action. This is not an entirely rigorous construction since one directly sums over the fixed loci in the absence

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of a rigorous intersection theory on an appropriate moduli space along the lines of [21]–[26]. In very concrete terms the computation reduces to a sum over Kontsevich graphs with involution. Nevertheless, we will show that the results obtained by this technique are in perfect agreement with large N duality and mirror symmetry predictions. The present approach is valid for all genera and in principle it does not require a large N dual description. We illustrate this method for a concrete toric model obtaining a generating functional which satisfies the integrality properties predicted in [19]. We also compare our results for $\mathbb{RP}^2$ instanton effects to a **B**-model local mirror computation obtaining precise agreement.

The paper is structured as follows. In section 2, we review some basic facts about topological orientifolds and discuss the basic principles of the method. Section 3 is devoted to concrete computations for a local conifold geometry, and in section 4 we treat a more involved toric model.

2. Orientifolds of topological A-models

We begin our analysis with some preliminary remarks on orientifolds of closed string topological **A**-models using [27]–[29] as basic references. In the following we will elaborate on instanton effects in the unoriented sector of the theory.

Let X be a toric noncompact Calabi-Yau threefold equipped with an antiholomorphic involution $I : X \to X$. Throughout this paper we will only consider freely acting involutions. The orientifold models are obtained by gauging a discrete symmetry of the form σI where $\sigma : \Sigma \to \Sigma$, therefore the orientifold group is $G = \{1, \sigma I\} \simeq \mathbb{Z}/2$. If Σ has genus zero, the action of σ is given by by $w \to -1/\bar{w}$ in terms of an affine coordinate w. The action of σ on surfaces of higher genus g is more intricate. The classification of antiholomorphic involutions of genus g surfaces goes back to Felix Klein [30, 31]. It is known that these involutions are completely characterized by three invariants $(g, n(\sigma), k(\sigma))$ where $n(\sigma)$ is the number of connected components of the fixed locus Σ_σ of σ, and $k(\sigma)$ is the index of orientability. The latter is defined to be two minus the number of connected components of $\Sigma \setminus \Sigma_\sigma$. Here we are interested in involutions of type $(g, 0, 1)$. In this case σ is an orientation reversing diffeomorphism of Σ so that the quotient $\Sigma/\langle\sigma\rangle$ is an unoriented surface without boundary. For future reference we will denote the cyclic group of order two $\langle\sigma\rangle$ by G_{ws}. Note that not all Riemann surfaces of a given genus admit such involutions. Those surfaces that admit antiholomorphic involutions are usually called symmetric Riemann surfaces. It is known that symmetric surfaces of type $(g, 0, 1)$ exist for any $g \geq 0$.

In order to construct the partition function of the theory, one has to sum over twisted sectors [27, 29], just as in ordinary orbifold theories. For a space-time orbifold theory (in which the action on the world-sheet is trivial) we have the well known formula

$$Z_g = \frac{1}{|G|^g} \sum_{\alpha \in \mathrm{Hom}(\pi_1(\Sigma), G)} Z_g(\alpha)\,. \tag{2.1}$$

The sum in the right hand side of (2.1) is over all representations of the fundamental group of Σ in the orbifold group G. In orientifold theories, we have a similar formula except that

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the fundamental group of Σ must be replaced by the orbifold fundamental group $\pi_1 O(\Sigma)$ defined with respect to the action of the world-sheet group G_{ws}. $\pi_1 O(\Sigma)$ is the group of all diffeomorphisms of the universal cover $\widetilde{\Sigma}$ of Σ which are lifts of elements of G_{ws}. For concrete applications, note that there is an exact sequence of the form

$$1 \to \pi_1(\Sigma) \to \pi_1 O(\Sigma) \to G_{ws} \to 1\,. \tag{2.2}$$

Then the twisted sectors of the theory are defined by representations $\alpha \in \mathrm{Hom}(\pi_1 O(\Sigma), G)$ [29] such that we have a commutative triangle diagram

$$\begin{tikzcd} \pi_1 O(\Sigma) \arrow[r, "\alpha"] \arrow[d] & G \arrow[dl] \\ G_{ws}. & \end{tikzcd} \tag{2.3}$$

The vertical arrow in the above diagram is the projection $\pi_1 O(\Sigma) \to G_{ws}$ of equation (2.1) with kernel $\pi_1(\Sigma) \subset \pi_1 O(\Sigma)$. Therefore the monodromy α is always trivial along the generators of $\pi_1(\Sigma)$. The map $G \to G_{ws}$ is defined by $\sigma I \to \sigma$. This means that there is only one twisted sector of the theory consisting of maps $f : \Sigma \to X$ satisfying the equivariance condition

$$f \circ \sigma = I \circ f\,. \tag{2.4}$$

So far these considerations are not specific to topological sigma models. In a topological theory, the partition function reduces to an instanton sum. The twisted sector instantons are area minimizing maps $f : \Sigma \to X$ satisfying the equivariance condition (2.4). Here the area must be defined with respect to a Kähler metric g on X left invariant by the involution $I : X \to X$. If g is an arbitrary Kähler metric on X, then $(g + I^* g)/2$ is an invariant metric in the same Kähler class. Since we are studying a topological **A**-model, the invariant metric need not be Calabi-Yau.

The equivariance condition (2.4) implies that f descends to a map between quotients $\widetilde{f} : \Sigma/G_{ws} \to X/I$. Since the involutions σ, I reverse orientation, the quotient spaces $\Sigma/G_{ws}, X/I$ do not have complex analytic structures. Instead they carry natural *dianalytic* structures [32]. Without giving too many details, a dianalytic structure is defined by an atlas (up to equivalence) whose transition functions on overlaps are either holomorphic or antiholomorphic. Then $\widetilde{f}$ is a dianalytic function if it is either holomorphic or antiholomorphic on each connected component of the domain. This means that in the unoriented theory we will have to sum over both holomorphic and antiholomorphic instantons. Since Σ/G_{ws} is connected, the two sums will be identical and the net effect is an overall factor of 2. Keeping this in mind, it suffices to consider only equivariant holomorphic maps.

In order to compute virtual instanton numbers we need to set up an intersection problem on an appropriate moduli space. From the previous paragraph, it follows that the moduli space in question should classify equivariant holomorphic maps up to a certain equivalence relation. Two such maps (Σ, f, σ), (Σ', f', σ') are said to be equivalent if there exists an isomorphism $\phi : \Sigma \to \Sigma'$ compatible with the antiholomorphic involutions so that $f = f' \circ \phi$. The compatibility condition for ϕ is very similar to (2.4)

$$\phi \circ \sigma = \sigma' \circ \phi\,. \tag{2.5}$$

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In principle one should give a rigorous construction of this moduli space and then define a virtual cycle of degree zero which counts the virtual instanton numbers. This program has been carried out for oriented holomorphic closed string maps in the context of Gromov-Witten theory [21]–[26]. Loosely speaking, in the present context one should construct an unoriented version of this theory.

However, this is not the route we will take in the present paper. Instead we will employ a semirigorous approach inspired from a similar treatment of open string Gromov-Witten theory [6, 7, 11]. To recall the basic points, note that in the open string framework one would like to count in appropriate sense holomorpic maps defined on bordered Riemann surfaces with boundary conditions specified by a lagrangian 3-cycle $L \subset X$. Although the moduli space of such maps is not rigorously understood, one can take a direct approach to this problem in the presence of a real torus action on X preserving L. The main point is that it is much easier to handle the fixed loci of the induced action on the moduli space instead of the moduli space itself. In particular one can give a computational definition of the virtual cycle using the localization formula of Graber and Pandharipande [23] adapted to the open string context. The main ingredient of this computation is the deformation complex of open string maps. One peculiar aspect of this approach is that the resulting virtual instanton numbers depend on the choice of toric weights. This dependence reflects the presence of a (real codimension one) boundary in the moduli space of maps, and it is partially understood in the context of large N duality. More details on this subject can be found in [7].

Our proposal is that a similar approach can be implemented for equivariant holomorphic maps as well. To begin with, we need a real torus action on X which is compatible with the antiholomorphic involution I. Since X is a toric noncompact threefold, it admits a T^3 action. Here we want to find a one dimensional subtorus $T \subset T^3$ which preserves the antiholomorphic involution I. Moreover, in order for the localization approach to be effective, the action of T should have only isolated fixed points on X. These conditions are somewhat restrictive, but we will show that such good actions exist in specific examples. Granting the existence of T, there is an induced action on the moduli space of equivariant holomorphic maps. The next step is to enumerate all fixed loci of the induced action. Then one can assign a local contribution to each connected component of the fixed locus using an equivariant version of the localization theorem of [23]. This approach is very similar to the original work of Kontsevich on localization and Gromov-Witten invariants [24]. We will show in the next section that the fixed loci are naturally classified in terms of Kontsevich graphs with involution. In order to compute the instanton numbers, one has to sum the local contributions over all fixed loci. Since we have not developed the fundamental theory, this procedure involves some subtle sign ambiguities which can be fixed by additional arguments. Essentially, one has to keep in mind that the final answer has to be a rational number independent of toric weights, if the moduli space is properly compactified. Here we will use a compactification of the moduli space inspired from [24, 25]. Namely we will allow the domain symmetric Riemann surfaces Σ to develop ordinary double points so that the resulting nodal surfaces admit freely acting antiholomorphic involutions. We must also impose a stability condition which makes the automorphism group of a symmetric map

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finite. Then we will sum over all fixed loci of the torus action satisfying these conditions, and the result should be a rational number. Note that there is an important difference between the case considered here and the open string maps studied in [7]. In the latter case, the compactified moduli space turns out to be a space with a real codimension one boundary. The boundary is associated to degenerations of the domain bordered surface in which nodes appear on boundary components. This is clearly a real codimension one phenomenon. In our case, such phenomena are absent because we consider only freely acting involutions, therefore the are no real codimension one fixed loci and no real codimension one phenomena. All these ideas will be made more concrete below.

3. Localization and invariant graphs I

In order to illustrate the basic principles we will first carry out the above program for a resolved conifold geometry. Exact results for topological unoriented amplitudes have been predicted in [19] using large N duality. Here we will show how these results can be reproduced by pure **A**-model computations. This example has also been considered in [15] in the context of mirror symmetry.

The toric threefold X is isomorphic to the total space of the rank two bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ over $\mathbb{P}^1$, which is given by the following toric data

$$\begin{array}{ccccc} & X_1 & X_2 & X_3 & X_4 \\ \mathbb{C}^* & 1 & 1 & -1 & -1\,. \end{array} \tag{3.1}$$

The second homology of X is generated by the zero section $C \subset X$, which is an isolated $(-1,-1)$ rational curve. Moreover, this is the only projective curve on X.

The freely acting antiholomorphic involution considered in [15] is given by

$$I : (X_1, X_2, X_3, X_4) \to \left(\overline{X}_2, -\overline{X}_1, \overline{X}_4, -\overline{X}_3\right). \tag{3.2}$$

Note that the quotient X/I is a smooth unorientable manifold which carries a dianalytic structure. The involution (3.2) preserves the zero section, therefore C/I is a dianalytic two-cycle on X/I with topology $\mathbb{RP}^2$. This cycle generates the second homology of X/I. For future reference we introduce local coordinates $z = \frac{X_1}{X_2}, u = X_2X_3, v = X_2X_4$ on the patch $\{X_2 \neq 0\} \subset X$. Then the involution (3.2) reads $\left(z,u,v\right) \to \left(-\frac{1}{\overline{z}}, -\overline{zv}, \overline{zu}\right)$.

As explained in the previous section, counting unoriented world-sheet instantons reduces to counting holomorphic maps $f : \Sigma \to X$ subject to the equivariance condition $f = I \circ f \circ \sigma$. Therefore at this point it may be helpful to recall some basic facts about the enumerative geometry of X. The main ingredients of Gromov-Witten can be found in [7]

a) a compact moduli space $\overline{M}_{g,0}(X,\beta)$ of stable maps to X with fixed genus g and fixed homology class $\beta = f_*[\Sigma] \in H_2(X)$,

b) a virtual fundamental cycle $[\overline{M}_{g,0}(X,\beta)]^{\mathrm{vir}}$ on the moduli space,

c) an orientation on the moduli space.

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If X is a Calabi-Yau threefold, the virtual cycle has dimension zero and the virtual number of maps is defined to be the degree of this cycle

$$C_{g,\beta} = \int_{[\overline{M}_{g,0}(X,\beta)]^{\rm vir}} 1\,. \tag{3.3}$$

This data gives rise to a truncated Gromov-Witten potential of the form

$$\mathcal{F}_X = \sum_{g\,\geq\,0}\ \sum_{\substack{\beta\in H_2(X)\\ \beta\neq 0}} C_{g,\beta}\, q^{\beta} \tag{3.4}$$

where q^β is a formal symbol satisfying $q^{\beta+\beta'} = q^\beta q^{\beta'}$.

In the present case, any holomorphic map $f : \Sigma \to X$ must factorize according to the following diagram

$$\begin{tikzcd} \Sigma \arrow[r, "f"] \arrow[dr] & X \\ & C. \arrow[u] \end{tikzcd} \tag{3.5}$$

This means that the moduli space of stable holomorphic maps $\overline{M}_{g,0}(X, d[C])$ is isomorphic to the moduli space of stable degree d maps to $\mathbb{P}^1$, $\overline{M}_{g,0}(\mathbb{P}^1, d)$. The virtual fundamental cycle can be constructed using a perfect obstruction complex on the moduli space [22, 26]. Moreover, there is a canonical orientation induced by the holomorphic structure.

The integral in the right hand side of (3.3) can be evaluated using the localization theorem of Graber and Pandharipande [23]. There is a $T = S^1$ action on X given by

$$e^{i\phi}\cdot (X_1, X_2, X_3, X_4) = (e^{i\lambda_1\phi}X_1, e^{i\lambda_2\phi}X_2, e^{i\lambda_3\phi}X_3, e^{i\lambda_4\phi}X_4) \tag{3.6}$$

which induces a T action on the moduli space $\overline{M}_{g,0}(\mathbb{P}^1, d)$.

Then one has a localization formula of the form

$$C_{g,d} = \sum_{\Xi} \int_{[\Xi]^{\rm vir}} \frac{1}{e_T(N_\Xi^{\rm vir})}\,, \tag{3.7}$$

where the sum is over all fixed loci of the torus action on the moduli space. In the right hand side $N_\Xi^{\rm vir}$ is the virtual normal bundle to Ξ and $[\Xi]^{\rm vir}$ is the virtual cycle on Ξ induced by the one on the ambient moduli space. e_T denotes the equivariant Euler class. The contributions of the fixed loci can be evaluated using the tangent obstruction complex, as explained in [23, 24]. Note that the individual terms in this sum take values in the fraction field of the representation ring of T, $\mathcal{R}_T$. Nevertheless the final result is a rational number as expected. Using these techniques, one can show that the Gromov-Witten potential (3.3) takes the following closed form [33]

$$\mathcal{F}_X = \sum_{d\geq 1} \frac{\mathrm{e}^{-dt}}{d\left(2\sin\frac{dg_s}{2}\right)^2}\,. \tag{3.8}$$

Here we have to solve a similar counting problem for holomorphic maps $f : \Sigma \to X$ satisfying the extra condition $f = I \circ f \circ \sigma$. Ideally, one would like to follow steps (a)–(c)

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outlined above, but this is not the route we will take here. Instead we will employ a less rigorous approach following similar ideas used in the context of open string enumerative geometry [6, 7, 11]. The main point is that in practice the fixed loci of a torus action on the moduli space are much easier to describe than the moduli space itself. Moreover, the contribution of each fixed locus to the right hand side of (3.7) can be easily evaluated using a local description of the perfect obstruction complex of the moduli space. Therefore, one could in principle obtain the final answer only from the data of the fixed loci supplemented with some local deformation theory. This is essentially the original approach proposed by Kontsevich in [24] where the fixed loci are classified in terms of graphs. Then one can design a simple algorithm for computing the contribution of a fixed locus in terms of the combinatorics of the associated graph.

In order to implement this method in our situation, we need a torus action on X compatible with the antiholomorphic involution. This will induce a torus action on the moduli space, and we can proceed with the classification of all fixed loci. The local contribution of each fixed locus will be evaluated using an equivariant version of the local tangent-obstruction complex. Finally, the virtual number of equivariant holomorphic maps will be obtained by summing over all fixed loci.

There are a couple of subtle points here which should be discussed in some detail. In Gromov-Witten theory one obtains a compact moduli space by including degenerate maps (Σ, f) subject to a stability condition. The domain Σ is allowed to degenerate to a reducible singular curve with only ordinary double points. Stability requires the map (Σ, f) to have a finite automorphism group. This means that any connected curve $\Sigma' \subset \Sigma$ which is mapped to a point must be stable in the sense of Deligne and Mumford [34].

Our problem is slightly more complicated since we have to classify triples (Σ, f, σ) where σ is an antiholomorphic involution of Σ of type $(g, 0, 1)$. For convenience, recall that two such triples are equivalent if there exists an isomorphism $\phi : \Sigma \longrightarrow \Sigma'$ such that $f = f' \circ \phi$ and $\sigma = \phi^{-1} \circ \sigma' \circ \phi$. Therefore in order to obtain a compact moduli space, we have to include degenerate maps with involution. In particular, the domain Σ of such a map must be a symmetric nodal Riemann surface which admits a antiholomorphic involution of type $(g, 0, 1)$. The classification of such objects is not a simple task, so the structure of the moduli space is not easy to understand. However, for our purposes it suffices to understand only the structure of the fixed loci Ξ, which is a more tractable question. The main outcome is that such fixed loci are classified in terms of Kontsevich graphs with involution. Let us discuss these aspects in more detail for the resolved conifold geometry.

Consider a generic torus action on X of the form (3.6). This action is compatible with the antiholomorphic involution (3.2) if the weights satisfy the conditions $\lambda_1 + \lambda_2 = 0$, $\lambda_3 + \lambda_4 = 0$. In terms of local coordinates (z, u, v), the T action reads

$$e^{i\phi} \cdot (z, u, v) = \left(e^{i\lambda_z} z, e^{i\lambda_u} u, e^{i\lambda_v} v\right), \tag{3.9}$$

where $\lambda_z = \lambda_1 - \lambda_2$, $\lambda_u = \lambda_1 + \lambda_3$, $\lambda_v = \lambda_1 + \lambda_4$. The compatibility condition becomes

$$\lambda_z + \lambda_u + \lambda_v = 0\,. \tag{3.10}$$

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If this condition is satisfied, there is an induced torus action on the moduli space of holomorphic maps with involution. In the following we describe the structure of the fixed loci.

Let us first recall the Kontsevich graph representation of fixed loci in $\overline{M}_{g,0}(X, d[C])$ [24]. To any invariant map $f : \Sigma \to X$ we can associate a connected graph Γ as follows. Let P_1, P_2 denote the fixed points of the torus action on X defined by

$$P_1: \quad X_1 = X_3 = X_4 = 0\,, \qquad P_2: \quad X_2 = X_3 = X_4 = 0\,. \tag{3.11}$$

Note that both P_1, P_2 lie on C.

i) The vertices $v \in V(\Gamma)$ represent connected components Σ_v of $f^{-1}(P_1, P_2)$, which can be either points or disjoint unions of several irreducible components of Σ. To each vertex v we associate a number $k_v \in \{1, 2\}$ defined by $f(\Sigma_v) = P_{k_v}$, and the arithmetic genus g_v of Σ_v.

ii) The edges $e \in E(\Gamma)$ correspond to irreducible components of Σ which are mapped onto C. By T-invariance, each such component Σ_e must be a rational curve, and $f|_{\Sigma_e} : \Sigma_e \to C$ must be a Galois cover of degree $d_e \geq 1$. In terms of local coordinates the restriction of f to Σ_e is given by $z = w^{d_e}$.

Then the claim of [24] is that the set of all fixed loci is in one to one correspondence with (equivalence classes of) graphs Γ subject to the following conditions

1. If $e \in E(\Gamma)$ is an edge connecting two vertices u, v, then $f(u) \neq f(v)$.

2. $1 - \chi(\Gamma) + \sum_{v \in V(\Gamma)} g_v = g$, where g_v is the arithmetic genus of the component Σ_v, and $\chi(\Gamma)$ is the Euler characteristic of Γ, $\chi(\Gamma) = |V(\Gamma)| - |E(\Gamma)|$.

3. $\sum_{e \in E(\Gamma)} d_e = d$.

For clarification, let us consider an invariant map of genus $g = 4$ and degree $d = 18$ as in figure 1. Note that each horizontal component maps to C with a certain degree as shown in the left figure. The corresponding marked graph is shown in the right figure. For each vertex v, the first integer represents the value of $f(v)$, i.e. the fixed point it maps to. The second integer is the arithmetic genus of Σ_v.

Now let us consider stable holomorphic maps with involution. Recall that the involution I acts on C by $[X_1 : X_2] \to [\overline{X}_2 : -\overline{X}_1]$, hence it exchanges the fixed points P_1, P_2. In local coordinates we have $z \to -1/\overline{z}$. The fixed loci consist of maps which satisfy $f = I \circ f \circ \sigma$. Therefore σ must send each component Σ_v of the domain of type $(1, g_v)$ to another component of type $\Sigma_{v'}$ of type $(2, g_{v'})$. In particular the two components must have the same genus $g_v = g_{v'}$. This means that Σ is symmetric under pairwise exchange of components Σ_v of type 1 and type 2. The corresponding graphs must accordingly be symmetric under pairwise exchange of vertices of type one and two.

The action of σ on the horizontal components can be of two types.

A) A component Σ_e may be mapped to itself by $w \to -1/\overline{w}$, where w is a local coordinate. Then the map $z = w^{d_e}$ is equivariant if and only if d_e is odd.

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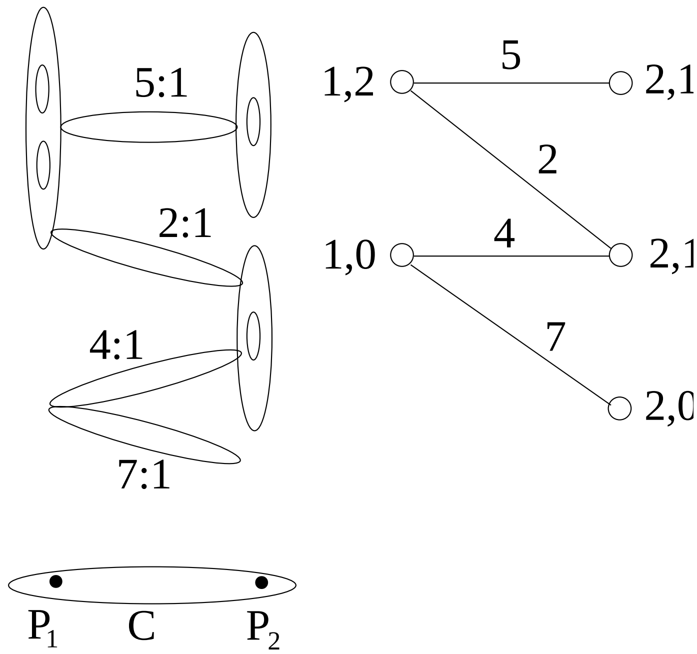

Figure 1: The Kontsevich graph associated to a fixed map of degree 18 and genus 4.

$B)$ A component Σ may be mapped to another component $\Sigma_{e'}$ of the same degree $d_e = d_{e'}$. In terms of local coordinates (w_1, w_2), the involution must be given by $w_1 \rightarrow (-1)^{d_e-1}/\overline{w}_2$, $w_2 \rightarrow (-1)^{d_e-1}/\overline{w}_1$. Then one can easily check that the map $z = w_1^{d_e}$, $z = w_2^{d_e}$ is equivariant.

In conclusion, the equivariance conditions impose important symmetry conditions on the fixed loci. These conditions are best summarized using the graph representation. The admissible graphs are characterized by the following conditions

(I) There is a involution $\tau_V : V(\Gamma) \rightarrow V(\Gamma)$ sending a vertex of type one to a vertex of type two, leaving the genus invariant $g_{\tau_V(v)} = g_v$.

(II) There is a involution $\tau_E : E(\Gamma) \rightarrow E(\Gamma)$ leaving the degree invariant, $d_{\tau_E(e)} = d_e$. If e is a fixed point of τ_E, d_e must be odd.

(III) The involutions τ_E, τ_V must be compatible in the following sense. If an edge e is attached to a vertex v, then $\tau_E(e)$ must be attached to $\tau_V(v)$.

Given such a graph Γ, we have to evaluate the local contribution of the associated fixed locus. Although we have not rigorously defined a virtual fundamental cycle, we will assume that such a construction exists and that a localization result of the type [23] holds in our case as well. Then the local contribution of a fixed locus can be computed using the tangent-obstruction complex of a holomorphic map with involution.

Recall that the tangent-obstruction complex of an arbitrary holomorphic map $f : \Sigma \rightarrow X$ is

$$0 \rightarrow \mathrm{Aut}(\Sigma) \rightarrow H^0(\Sigma, f^*T_X) \rightarrow \mathbb{T}^1 \rightarrow \mathrm{Def}(\Sigma) \rightarrow H^1(\Sigma, f^*T_X) \rightarrow \mathbb{T}^2 \rightarrow 0\,, \qquad (3.12)$$

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where $\mathbb{T}^1$, $\mathbb{T}^2$ are the deformation and respectively obstruction space of (Σ, f). This can be interpreted as a complex of sheaves on the moduli space. If (Σ, f) represents a point of a fixed locus Ξ, the virtual normal bundle to Ξ is given by the moving part of the above complex with respect to the torus action [23]. The fixed part determines the induced virtual cycle $[\Xi]^{\mathrm{vir}}$. The local contribution of a fixed locus represented by a graph Γ can be evaluated using normalization exact sequences [23, 24]. In order to write down the answer in concise from, let us define a flag [24] to be pair $(v, e) \in V(\Gamma) \times E(\Gamma)$ so that v lies on e. One can think of a flag as an oriented edge. We also define the valence of a vertex v to be the number of flags (v, e). Recall that a vertex represents either a point of Σ or a connected union of irreducible components mapping to one of the fixed points $P_1, P_2 \in X$. If v represents a point p_v we have either $val(v) = 1$, if p_v is a smooth point or $val(v) = 2$ if p_v is an ordinary double point. We will denote the set of all such vertices by $V_{02}(\Gamma)$. Note that in the last case, p_v is the transverse intersection point of two horizontal components $\Sigma_{e_1(v)}, \Sigma_{e_2(v)}$ corresponding to the flags $(v, e_1(v)), (v, e_2(v))$. In all other cases, v represents a stable genus g_v curve Σ_v with $val(v)$ marked points $(p_1, \ldots, p_{val(v)})$. The markings are points of intersection of Σ_v with horizontal components $\Sigma_{e_k(v)}$ where $(v, e_k(v))$, $k = 1, \ldots, val(v)$ are all flags containing v. The fixed locus Ξ is isomorphic to a quotient of the direct product $\prod_{v \in V(\Gamma)\setminus V_{02}(\Gamma)} \overline{M}_{g_v, val(v)}$ by a finite group G. G is the automorphism group of an arbitrary fixed map in Ξ and it fits in the exact sequence

$$1 \longrightarrow \prod_{e \in E(\Gamma)} \mathbb{Z}/d_e \longrightarrow G \longrightarrow \mathrm{Aut}(\Gamma) \longrightarrow 1, \tag{3.13}$$

where $Aut(\Gamma)$ is the automorphism group of the graph Γ. Note that each marked point p_k determines a Mumford class $\psi_k \in H^*(\overline{M}_{g_v, val(v)})$.

We have a normalization exact sequence of the form

$$0 \to f^*T_X \to \bigoplus_{e \in E(\Gamma)} f_e^* T_X \oplus \bigoplus_{v \in V(\Gamma)} f_v^* T_X \to \bigoplus_{v \in V(\Gamma)} (T_{P_{k_v}} X)^{val(v)} \to 0\,. \tag{3.14}$$

The associated long exact sequence reads

$$\begin{aligned} 0 \to H^0(\Sigma, f^*T_X) &\to \bigoplus_{e \in E(\Gamma)} H^0(\Sigma_e, f_e^* T_X) \oplus \bigoplus_{v \in V(\Gamma)} T_{P_{k_v}} X \to \bigoplus_{v \in V(\Gamma)} (T_{P_{k_v}} X)^{val(v)} \\ \to H^1(\Sigma, f^*T_X) &\to \bigoplus_{e \in E(\Gamma)} H^1(\Sigma_e, f_e^* T_X) \oplus \bigoplus_{v \in V(\Gamma)} H^1(\Sigma_v, \mathcal{O}_{\Sigma_v}) \otimes T_{P_{k_v}} X \to 0\,. \end{aligned} \tag{3.15}$$

The moving part of the automorphism group $Aut(\Sigma)$ consists of holomorphic vector fields on the horizontal components Σ_e which vanish at the points of Σ_e which are nodes of Σ. let ν_e denote the divisor of such points on Σ_e. Then we have

$$\mathrm{Aut}(\Sigma)^m = \oplus_{e \in E(\Gamma)} H^0(\Sigma_e, T_{\Sigma_e}(-\nu_e))\,. \tag{3.16}$$

The moving part of $Def(\Sigma)$ consists only of deformations of the nodes which lie at least

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on one horizontal component. That is we have

$$\mathrm{Def}(\Sigma)^m = \bigoplus_{\substack{v\in V_{02}(\Gamma)\\ val(v)=2}} T_{p_v}(\Sigma_{e_1(v)})\otimes T_{p_v}(\Sigma_{e_2(v)}) \oplus \bigoplus_{v\in V(\Gamma)\setminus V_{02}(\Gamma)} \left(\oplus_{k=1}^{val(v)} T_{p_k}\Sigma_{e_k(v)}\otimes T_{p_k}\Sigma_v\right). \tag{3.17}$$

Therefore the T-equivariant K-theory class of the virtual normal bundle to Ξ is given by

$$\begin{aligned}
[N_\Xi^{\mathrm{vir}}] = &\sum_{e\in E(\Gamma)} \left(-[H^1(\Sigma_e, f^*T_X)^m] + [H^0(\Sigma_e, f^*T_X)^m] - [H^0(\Sigma_e, T_{\Sigma_e})^m]\right) + \\
&+ \sum_{\substack{v\in V_{02}(\Gamma)\\ val(v)=2}} \left(-[T_{P_{k_v}}X] + [T_{p_v}\Sigma_{e_1(v)}] + [T_{p_v}\Sigma_{e_2(v)}] + [T_{p_v}\Sigma_{e_1(v)}\otimes T_{p_v}\Sigma_{e_2(v)}]\right) + \\
&+ \sum_{v\in V(\Gamma)\setminus V_{02}(\Gamma)} \left(-[H^1(\Sigma_v, f^*T_X)] + \sum_{k=1}^{val(v)} [T_{p_k}\Sigma_v\otimes T_{p_k}\Sigma_{e_k(v)}] + \sum_{k=1}^{val(v)} [T_{p_k}\Sigma_{e_k(v)}]\right) - \\
&- \sum_{v\in V(\Gamma)\setminus V_{02}(\Gamma)} \left((val(v)-1)[T_{P_{k_v}}X]\right).
\end{aligned} \tag{3.18}$$

The local contribution of Ξ is of the form

$$\int_{[\Xi]^{\mathrm{vir}}} \frac{1}{e_T(N_\Xi^{\mathrm{vir}})} = \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{e\in E(\Gamma)} F(e) \prod_{v\in V_{02}(\Gamma)} G(v) \prod_{v\in V(\Gamma)\setminus V_{02}(\Gamma)} \int_{(\overline{M}_{g_v,val(v)})_T} H(v), \tag{3.19}$$

where the expressions $F(e), G(v), H(v)$ are given as follows. For any vertex v we denote by $\rho_{k_v}^{1,2,3}$ the weights of the torus action on the holomorphic tangent space to X at P_{k_v}. Hence, for $k_v = 1$ we have $(\rho_1^1, \rho_1^2, \rho_1^3) = (-\lambda_z, -\lambda_u, -\lambda_v)$ and for $k_v = 2$, $(\rho_2^1, \rho_2^2, \rho_2^3) = (\lambda_z, -\lambda_u-\lambda_z, -\lambda_v-\lambda_z)$. We have

$$\begin{aligned}
F(e) &= \frac{(-1)^{d_e-1}}{d_e(d_e!)^2} \prod_{k=1}^{d_e-1} \left(k + \frac{d_e\lambda_u}{\lambda_z}\right)\left(k+\frac{d_e\lambda_v}{\lambda_z}\right) \\
G(v) &= \begin{cases} 1 & \text{if } v\in V_{02}(\Gamma),\ val(v)=1 \\ \dfrac{d_{e_1}^2 d_{e_2}^2 \rho_{k_v}^1\rho_{k_v}^2\rho_{k_v}^3}{(d_{e_1}+d_{e_2})(\rho_{k_v}^1)^3} & \text{if } v\in V_{02}(\Gamma),\ val(v)=2 \end{cases} \\
H(v) &= \left(\rho_{k_v}^1\rho_{k_v}^2\rho_{k_v}^3\right)^{val(v)-1} \frac{c_{g_v}(\mathbb{E}^\vee(\rho_{k_v}^1 H))c_{g_v}(\mathbb{E}^\vee(\rho_{k_v}^2 H))c_{g_v}(\mathbb{E}^\vee(\rho_{k_v}^3 H))}{\prod_{k=1}^{val(v)} \frac{\rho_{k_v}^1 H}{d_{e_k}}\left(-\psi_k + \frac{\rho_{k_v}^1 H}{d_{e_k}}\right)},
\end{aligned} \tag{3.20}$$

where $\mathbb{E}$ is the Hodge bundle on the moduli space $\overline{M}_{g,val(v)}$, and $H\in H_T^*(pt)$ generates the equivariant cohomology of a point.

Our goal is to derive a similar formula for a fixed locus in the moduli space of equivariant holomorphic maps. Suppose we are given a fixed locus Ξ_s in the moduli space of symmetric holomorphic maps described by a symmetric graph Γ_s. Recall that Γ_s is endowed with a pair of involutions $\tau_E : E(\Gamma_s)\to E(\Gamma_s)$, $\tau_V : V(\Gamma_s)\to V(\Gamma_s)$ which satisfy conditions $(I)--(III)$ above. We can regard τ_E, τ_V as generators of cyclic groups of order two $\langle\tau_E\rangle$, $\langle\tau_V\rangle$ acting on $E(\Gamma)$ and respectively $V(\Gamma)$.

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According to our previous discussion, $\sigma : \Sigma \to \Sigma$ maps each component $\Sigma_v \in f^{-1}(P_1)$ to the component $\Sigma_{\tau_V(v)} \in f^{-1}(P_2)$. This map is an antiholomorphic isomorphism between Σ_v and $\Sigma_{\tau_V(v)}$. Any deformation of the triple (Σ, f, σ) must preserve this relation, therefore the fixed locus Ξ_s is isomorphic to a quotient by a finite group of the direct product $\prod_{<v>\in V(\Gamma_s)/\langle\tau_V\rangle} \overline{M}_{v,val(v)}$. The notation means that we take a single factor $\overline{M}_{v,val(v)}$ for each orbit of the group $\langle\tau_V\rangle$ in $V(\Gamma_s)$.

Next we have to determine the virtual normal bundle to Ξ_s. The main point is that we have to take into account only infinitesimal deformations of (Σ, f, σ) which preserve the symmetric structure and the equivariance condition $f = I \circ f \circ \sigma$. This will impose constraints on the obstructions as well. An important observation is that σ induces antiholomorphic involutions on the cohomology groups $H^*(\Sigma, T_\Sigma)$. Similarly, the pair (I, σ) induces antiholomorphic involutions on $H^*(\Sigma, f^*T_\Sigma)$. The tangent-obstruction complex of a symmetric map (Σ, f, σ) can be obtained by taking the invariant part of (3.12)

$$0 \to H^0(\Sigma, T_\Sigma)^\sigma \to H^0(\Sigma, f^*T_X)^{(I,\sigma)} \to \mathbb{T}^1_s \to H^1(\Sigma, T_\Sigma)^\sigma \to H^1(\Sigma, f^*T_X)^{(I,\sigma)} \to \mathbb{T}^2_s \to 0\,. \tag{3.21}$$

In order to determine the virtual normal bundle, we have to take the moving part of (3.21). This is equivalent to taking the invariant part of the moving part of (3.12). Let us consider the action of the antiholomorphic involution on all cohomology groups involved in (3.15), (3.16) and (3.17). Recall that σ permutes the components of Σ according to the rules $(I) - -(III)$ below figure 1. In particular, a horizontal component Σ_e is mapped either to itself or to another horizontal component of $\Sigma_{\tau_E(e)}$ equal degree. A vertical component Σ_v is mapped to another vertical component $\Sigma_{\tau_V(v)}$ of the same genus.

If $\tau_E(e) \neq e$, σ induces an antiholomorphic involution on

$$H^0(\Sigma_e, T_{\Sigma_e}(-\nu_e)) \oplus H^0(\Sigma_{\tau_E(e)}, T_{\Sigma_{\tau_E(e)}}(-\nu_e))$$

and we are instructed to take the invariant real subspace. This can be (non-canonically) identified with $H^0(\Sigma_e, T_{\Sigma_e}(-\nu_e))$ as a complex vector space. Note that $H^0(\Sigma_e, T_{\Sigma_e}(-\nu_e))$ and $H^0(\Sigma_{\tau_E(e)}, T_{\Sigma_{\tau_E(e)}}(-\nu_e))$ are isomorphic as T-equivariant complex vector spaces, hence $H^0(\Sigma_e, T_{\Sigma_e}(-\nu_e))$ depends only on the orbit $\langle e\rangle \in E(\Gamma)/\langle\tau_E\rangle$. However, there still is an ambiguity in this process related to the choice of an orientation on the moduli space. As noticed before, in standard Gromov-Witten theory, the moduli spaces come equipped with a canonical orientation induced by the complex structure. Here, the fixed subspace does not carry a canonical complex structure. Therefore we could equally well identify it to the complex conjugate $H^0(\Sigma_e(-\nu_e))^*$, and this gives rise to a sign ambiguity in the evaluation of the contribution of the fixed locus. This ambiguity cannot be resolved in the absence of a rigorous construction of the moduli space and the virtual cycle. Similar problems will be encountered throughout the rest of this analysis as well. The only available solution is to fix a set of conventions and keep in mind that the resulting contributions may be off by a sign, which will be fixed later.

If $\tau_E(e) = e$, there is an antiholomorphic involution $H^0(\Sigma_e, T_{\Sigma_e}(-\nu_e)) \to H^0(\Sigma, T_{\Sigma_e}$ $(-\nu_e))$, and we have to take the fixed subspace. The details of this computation are given in appendix A.

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The action of σ on the deformations (3.17) can be easily inferred from the action on Σ, which exchanges the nodes pairwise. For any two nodes p, p' forming an orbit of σ the invariant deformations can be noncanonically identified to the deformation of p (up to a complex conjugation).

Similar considerations apply to the cohomology groups $H^{0,1}(\Sigma_e, f_e^* T_X)$ and $H^{0,1}(\Sigma_v, f_v^* T_X)$. The former are pairwise identified if $\tau_E(e) \neq e$, by analogy with the automorphism groups. For $\tau_E(e) = e$, the invariant subspaces are computed again in appendix A. The groups $H^{0,1}(\Sigma_v, f_v^* T_X)$ are also pairwise identified. Given an orbit $(v, \tau_V(v))$, we can choose the invariant subspace to be $H^{0,1}(\Sigma_v, f_v^* T_X)$ with $k_v = 1$. Collecting the facts, the equivariant K-theory class of the virtual normal bundle $N^{\rm vir}_{\Xi_s}$ can be written as follows

$$
\begin{aligned}
[N^{\rm vir}_{\Xi_s}] = & \sum_{\substack{\langle e\rangle \in E(\Gamma)/\langle \tau_E\rangle \\ \tau_E(e)\neq e}} \Big(-[H^1(\Sigma_e, f_e^*T_X)^m] + [H^0(\Sigma_e, f_e^*T_X)^m] - [H^0(\Sigma_e, T_{\Sigma_e})^m]\Big) + \\
& + \sum_{\substack{e\in E(\Gamma) \\ \tau_E(e)=e}} \Big(-[H^1(\Sigma_e, f_e^*T_X)^{\sigma,m}] + [H^0(\Sigma_e, f_e^*T_X)^{\sigma,m}] - [H^0(\Sigma_e, T_{\Sigma_e})^{\sigma,m}]\Big) + \\
& + \sum_{\substack{v\in V_{02}(\Gamma) \\ val(v)=2, k_v=1}} \Big(-[T_{P_{k_v}}X] + [T_{p_v}\Sigma_{e_1(v)}] + [T_{p_v}\Sigma_{e_2(v)}] + [T_{p_v}\Sigma_{e_1(v)} \otimes T_{p_v}\Sigma_{e_2(v)}]\Big) + \\
& + \sum_{v\in V(\Gamma)\setminus V_{02}(\Gamma), k_v=1} \left(-[H^1(\Sigma_v, f^*T_X)] + \sum_{k=1}^{val(v)} [T_{p_k}\Sigma_v \otimes T_{p_k}\Sigma_{e_k(v)}] + \sum_{k=1}^{val(v)} [T_{p_k}\Sigma_{e_k(v)}]\right) - \\
& - \sum_{v\in V(\Gamma)\setminus V_{02}(\Gamma), k_v=1} \Big((val(v)-1)[T_{P_{k_v}}X]\Big)\,. \qquad (3.22)
\end{aligned}
$$

The automorphism group G_s is an extension given by the exact sequence

$$
1 \longrightarrow \prod_{e\in E(\Gamma)/\langle \tau_E\rangle} \mathbb{Z}/d_e \longrightarrow G_s \longrightarrow \mathrm{Aut}(\Gamma_s) \longrightarrow 1, \qquad (3.23)
$$

where $Aut(\Gamma_s)$ is the group of automorphisms which preserve the involution.

The local contribution of the symmetric fixed locus Ξ_s reads

$$
\begin{aligned}
\int_{[\Xi_s]^{\rm vir}} \frac{1}{e_T(N^{\rm vir}_{\Xi_s})} = & \frac{1}{|\mathrm{Aut}(\Gamma_s)|} \prod_{\substack{\langle e\rangle\in E(\Gamma)/\langle\tau_E\rangle \\ \tau_E(e)\neq e}} F(e) \prod_{\substack{e\in E(\Gamma) \\ \tau_E(e)=e}} C(e) \times \\
& \times \prod_{\substack{v\in V_{02}(\Gamma) \\ k_v=1}} G(v) \prod_{\substack{v\in V(\Gamma)\setminus V_{02}(\Gamma) \\ k_v=1}} \int_{(\overline{M}_{g_v, val(v)})_T} H(v)\,, \qquad (3.24)
\end{aligned}
$$

where $F(e), G(e), H(e)$ are given in (3.20). The function $C(e)$ is derived in appendix A

$$
C(e) = \frac{1}{d_e(d_e!)} \prod_{k=1}^{d_e-1} \left(k + \frac{d_e \lambda_v}{\lambda_z}\right). \qquad (3.25)
$$

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3.1 Computations

Let us now perform some concrete computations and run a comparison test with large N duality predictions. First note that on general grounds, the unoriented topological free energy for the model considered here has an expansion of the form

$$\mathcal{F}_{(X,I)} = \sum_{h\geq 0}\sum_{c\geq 1} g_s^{-\chi} \sum_{d\geq 1} C_{-\chi,d}\, q^{d/2} \,. \tag{3.26}$$

where χ denotes the Euler characteristic of the unoriented Riemann surface $\Sigma/\langle\sigma\rangle$. We have $-\chi = 2h-2+c$ where h is the number of handles and c is the number of crosscaps. If the covering surface Σ has genus g, we have $2h-2+c = g-1$. Moreover, $d\geq 1$ denotes the degree of the map $\widetilde{f} : \Sigma/\langle\sigma\rangle \to X/I$. Note that the coefficients $C_{\chi,d}$ depend only on χ and not (h,c) taken separately. This is commonly referred to in the physics literature as trading a certain number of crosscaps for a certain number of handles. In fact we will show below that the fixed loci in the moduli space of symmetric maps of genus g may have different values of (h,c) as long as the combination $2h-2+c$ stays the same.

Now, recall that large N duality [19] predicts an exact formula for all genus unoriented topological amplitudes of the form

$$\mathcal{F}_{(X,I)} = \sum_{\substack{d\geq 1\\ d\ \text{odd}}} \frac{q^{d/2}}{2n\sin\frac{dg_s}{2}} \,. \tag{3.27}$$

Using the generating functional for modified Bernoulli numbers

$$\frac{t/2}{\sin(t/2)} = \sum_{k=0}^{\infty} b_k t^{2k} \,, \tag{3.28}$$

we can rewrite (3.27) as

$$\mathcal{F}_{(X,I)} = \sum_{\substack{d\geq 1\\ d\ \text{odd}}}\sum_{h\geq 0}\sum_{\substack{c\geq 1\\ c\ \text{odd}}} g_s^{2h-2+c} d^{2h-3+c} b_{h+\frac{c-1}{2}} q^{d/2} \,. \tag{3.29}$$

This allows us to identify all the coefficients $C_{\chi,d}$ in (3.26). Our goal is to reproduce these results using the **A**-model approach developed so far. We start with $\mathbb{RP}^2$ instantons, that is $\chi=-1$ and consider several values of d.

i) $(\chi,d) = (-1,1)$. There is a single fixed locus represented by a graph with one edge of degree $d_e = 1$. This can be trivially evaluated to $C_{-1,1} = 1$, which is the correct result.

ii) $(\chi,d) = (-1,3)$. There are two fixed loci represented by the symmetric graphs in figure 2. The graphs in the first row are symmetric graphs as discussed above. In case (a) we have an involution mapping the horizontal component onto itself. In case (b) the involution exchanges the outer horizontal components and maps the middle component onto itself. This information can be conveniently encoded in the quotient

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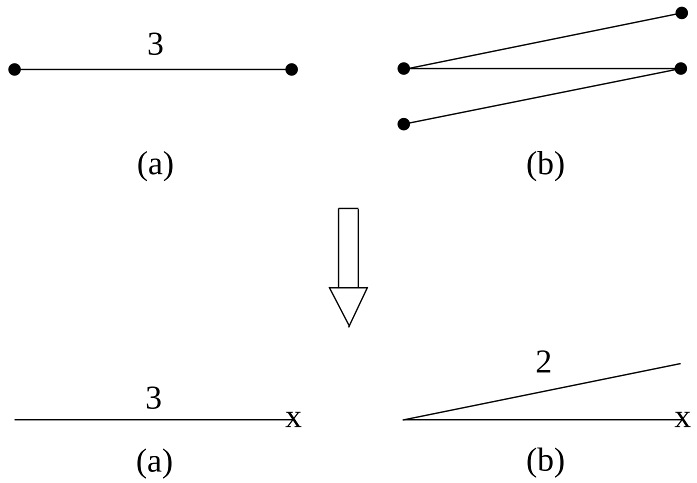

Figure 2: Degree 3 symmetric graphs.

graphs represented on the second row. Horizontal components with an $\mathbf{x}$ attached to a vertex represent odd multicovers of $\mathbb{RP}^2 \to \mathbb{RP}^2$. The other horizontal components correspond to even multicovers of $\mathbb{P}^1 \to \mathbb{RP}^2$. The local contributions are

$$C^{(a)}_{-1,3} = \frac{1}{18}\left(1+\frac{3\lambda_v}{\lambda_z}\right)\left(2+\frac{3\lambda_v}{\lambda_z}\right), \qquad C^{(b)}_{-1,3} = \frac{1}{2}\frac{\lambda_u\lambda_v}{\lambda_z^2}. \tag{3.30}$$

Taking into account the relation

$$\lambda_z + \lambda_u + \lambda_v = 0\,, \tag{3.31}$$

it is straightforward to check that

$$C_{-1,3} = C^{(a)}_{-1,3} + C^{(b)}_{-1,3} = \frac{1}{9}\,, \tag{3.32}$$

which matches the large N duality result. Note that this is true for any torus action compatible with the antiholomorphic involution. As opposed to open string localization computations, the result is independent of the toric weights, which reflects the compactness of the moduli space.

iii) $(\chi, d) = (-1, 5)$. Here we find five symmetric graphs of total degree five represented (together with their quotient graphs) in figure 3.

The local contributions are

$$\begin{aligned} C^{(a)}_{-1,5} &= \frac{1}{600}\frac{24\lambda_u^4 - 154\lambda_u^3\lambda_v + 269\lambda_v^2\lambda_u^2 - 154\lambda_u\lambda_v^3 + 24\lambda_v^4}{\lambda_z^4}\,, \\ C^{(c)}_{-1,5} &= \frac{1}{6}\frac{\lambda_u\lambda_v(\lambda_u-\lambda_v)^2}{\lambda_z^4}\,, \\ C^{(b)}_{-1,5} &= \frac{1}{8}\frac{\lambda_u\lambda_v(2\lambda_u^2 - 5\lambda_u\lambda_v + 2\lambda_v^2)}{\lambda_z^4}\,, \qquad C^{(d)}_{-1,5} = \frac{1}{4}\frac{\lambda_u^2\lambda_v^4}{\lambda_z^4}\,, \qquad C^{(e)}_{-1,5} = \frac{1}{2}\frac{\lambda_u^2\lambda_v^4}{\lambda_z^4}\,. \end{aligned} \tag{3.33}$$

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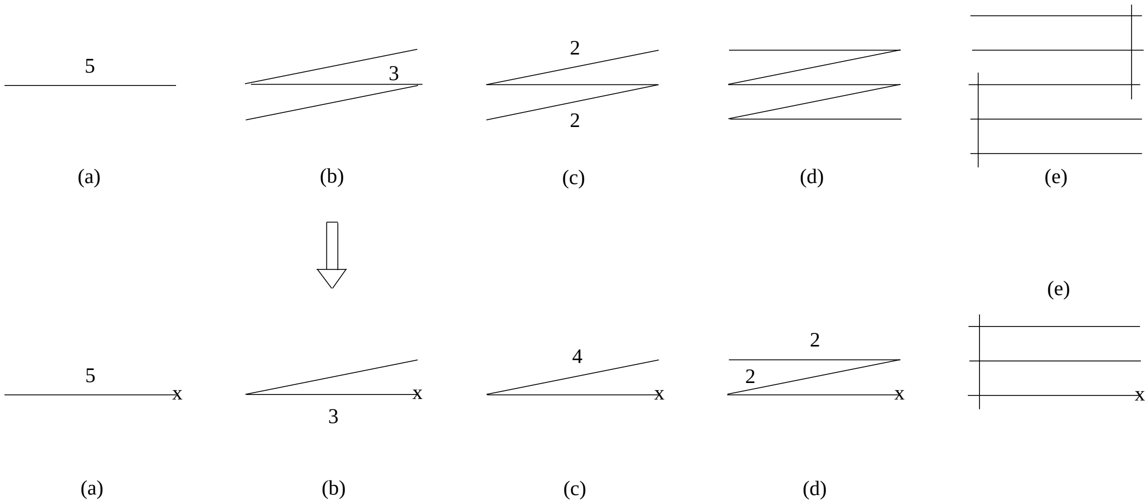

Figure 3: Degree 5 symmetric graphs.

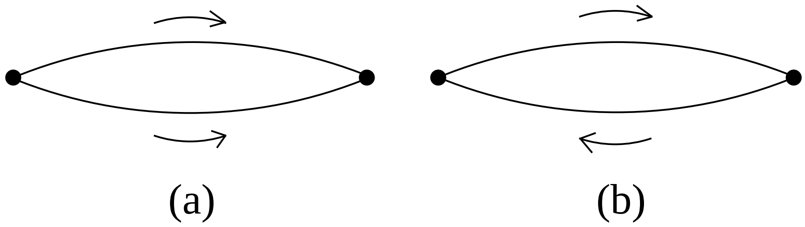

Figure 4: $(\chi, d) = (0, 2)$ symmetric graphs.

Using the condition (3.31), we obtain

$$C_{-1,5} = C^{(a)}_{-1,5} + \cdots + C^{(e)}_{-1,5} = \frac{1}{25}\,. \tag{3.34}$$

This result is again independent of the choice of a torus action which preserves the involution.

The higher degree computations become more cumbersome because the number of graphs increases rapidly. For example at degree seven, there are thirteen symmetric graphs which yield $C_{-1,7} = 1/49$ as expected. Although we do not have a general proof for any degree (at least for arbitrary values of torus weights subject to (3.6)), this is strong supporting evidence for our approach. The computation simplifies dramatically by making a special choice of torus weights. However, before discussing this option, it may be instructive to perform higher genus computations for arbitrary weights. Note that if $(\chi, d) = (0, 1)$, $C_{\chi,d}$ is trivially zero.

iv) $(\chi, d) = (0, 2)$. This is an interesting case. Topologically, we have a single symmetric graph with two edges of degree one and two vertices as shown in figure 4.

However, we should remember at this point that the moduli space classifies triples (Σ, f, σ) up to equivalence. In the present case, Σ is supposed to be a prestable

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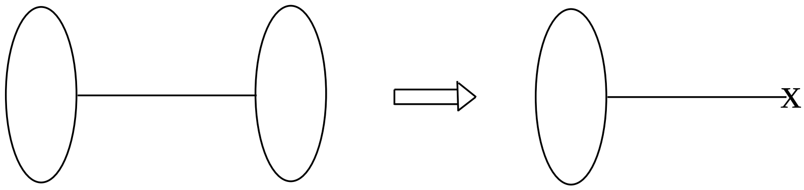

Figure 5: $(\chi, d) = (1, 1)$ symmetric graphs.

curve of genus $g = 1$, and $\sigma : \Sigma \to \Sigma$ a freely acting antiholomorphic involution. In order to obtain a fixed map under the torus action, Σ must degenerate to a nodal curve of arithmetic genus one as in figure 4. We claim that this curve admits two inequivalent antiholomorphic involutions. Let w_1, w_2 denote affine coordinates on the two components such that the restrictions of f to the two components are $z = w_1$, $z = w_2$. One can define the following antiholomorphic involutions

$$\begin{aligned} \sigma_1 :\quad & w_1 \to -\frac{1}{\overline{w}_1}, \qquad w_2 \to -\frac{1}{\overline{w}_2} \\ \sigma_2 :\quad & w_1 \to -\frac{1}{\overline{w}_2}, \qquad w_2 \to -\frac{1}{\overline{w}_1}. \end{aligned} \tag{3.35}$$

The domain Σ has an automorphism $\phi : \Sigma \to \Sigma$ which exchanges the two components. One can check that ϕ preserves both involutions σ_1, σ_2, hence the triples (Σ, f, σ_1), (Σ, f, σ_2) are not equivalent. They represent distinct fixed points of the moduli space, each having an automorphism group of order two. Both local contributions are equal to $\frac{\lambda_u\lambda_v}{4\lambda_z^2}$ up to a sign ambiguity, which has been discussed in the previous section. We do not know how to fix this ambiguity from the first principles, so we have to rely on duality predictions and the compactness assumption. Large N duality predicts a zero result for this amplitude. Moreover, by compactness of the moduli space, the result has to be independent of toric weights. This can be achieved only if the two contributions have a relative minus sign, in which case the result is trivial. We will simply adopt this rule as part of our computational definition of the virtual cycle. Additional supporting evidence will be found below.

v) $(\chi, d) = (1, 1)$. We have $2h - 2 + c = 1$, and $c = 1$, since the degree is one. This implies that $h = 1$, and there is a single graph represented in figure 5. Note that we have represented the genus one vertex by an ellipse in order to facilitate an intuitive understanding of the invariant map. Similarly, genus zero vertices will be represented by a vertical line segment in the following. The corresponding invariant is

$$\begin{aligned} C_{1,1} &= \frac{1}{(-\lambda_z H)} \int_{\overline{M}_{1,1}} \frac{c_1(\mathbb{E}^\vee(-\lambda_z H))c_1(\mathbb{E}^\vee(-\lambda_u H))c_1(\mathbb{E}^\vee(-\lambda_v H))}{(-\psi_1 - \lambda_z H)} \\ &= \frac{\lambda_u\lambda_v}{\lambda_z^2}\int_{\overline{M}_{1,1}} \psi - \frac{\lambda_z\lambda_u + \lambda_u\lambda_v + \lambda_v\lambda_z}{\lambda_z^2}\int_{\overline{M}_{1,1}} \lambda \\ &= b_1\,. \end{aligned} \tag{3.36}$$

Note that at the last step we had to use again the condition (3.31). This is the expected result.

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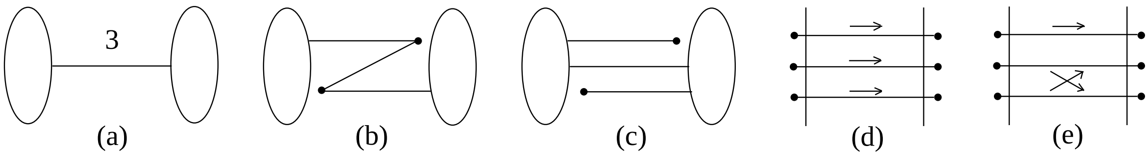

Figure 6: $(\chi, d) = (1,3)$ symmetric graphs.

vi) $(\chi, d) = (1,3)$. This is a more interesting case. We have $2h-2=1$ and $d=3$ which can be realized either as $(h,c)=(1,1)$ or as $(h,c)=(0,3)$. Therefore we can have surfaces with one handle and one crosscap or surfaces with no handles, but three crosscaps, as shown in figure 6.

The local contributions of these fixed loci read

$$
\begin{aligned}
C_{1,3}^{(a)} &= -\frac{b_1}{2}\frac{5\lambda_z^2\lambda_u\lambda_v - 2\lambda_z^4 + 18\lambda_v^2\lambda_u^2}{\lambda_z^4}\,, & C_{1,3}^{(b)} &= \frac{b_1}{2}\frac{\lambda_u\lambda_v}{\lambda_z^2}\,,\\
C_{1,3}^{(c)} &= b_1\frac{\lambda_u\lambda_v(\lambda_u\lambda_v - 2\lambda_u\lambda_z - 2\lambda_v\lambda_z)}{\lambda_z^4}\,, & C_{1,3}^{(d)} &= -\frac{1}{6}\frac{\lambda_u^2\lambda_v^2}{\lambda_z^4}\,,\\
C_{1,3}^{(e)} &= \frac{1}{2}\frac{\lambda_u^2\lambda_v^2}{\lambda_z^4}\,.
\end{aligned}
\tag{3.37}
$$

Some comments are in order here. In the process of evaluating graphs (a), (b) and (c) one has to integrate polynomials in tautological and Hodge classes on $\overline{M}_{1,1}$ and $\overline{M}_{1,2}$. This can be done using string and dilaton equations as explained for example in [35]. We omit the details since this is standard material. Graphs (d) and (e) are more subtle. It is clear that the domain Σ is the same in both cases. However, we obtain again two distinct fixed points because we can define inequivalent involutions. Let w_1, w_2, w_3 denote local affine coordinates on the three horizontal components so that the map f is given by $z=w_1,\ z=w_2,\ z=w_3$. Note that the automorphism group of Σ is isomoprhic to S_3 (the permutation group of three letters). First, we have an antiholomorphic involution

$$
\sigma_1: \quad w_1 \to -\frac{1}{\overline{w}_1}\,, \qquad w_2 \to -\frac{1}{\overline{w}_2}\,, \qquad w_3 \to -\frac{1}{\overline{w}_3}
\tag{3.38}
$$

which is preserved by the whole automoprhism group S_3. However, one can define three more antiholomorphic involutions

$$
\begin{aligned}
\sigma_2: &\quad w_1 \to -\frac{1}{\overline{w}_1}\,, & w_2 &\to -\frac{1}{\overline{w}_3}\,, & w_3 &\to -\frac{1}{\overline{w}_2}\\
\sigma_3: &\quad w_1 \to -\frac{1}{\overline{w}_3}\,, & w_2 &\to -\frac{1}{\overline{w}_2}\,, & w_3 &\to -\frac{1}{\overline{w}_1}\\
\sigma_4: &\quad w_1 \to -\frac{1}{\overline{w}_2}\,, & w_2 &\to -\frac{1}{\overline{w}_1}\,, & w_3 &\to -\frac{1}{\overline{w}_3}\,.
\end{aligned}
\tag{3.39}
$$

Writing S_3 as $\mathbb{Z}_2 \times \mathbb{Z}_3$, one can check that each such involution is preserved by the $\mathbb{Z}_2$ subgroup, and $(\sigma_2, \sigma_3, \sigma_4)$ form an orbit of the $\mathbb{Z}_3$ subgroup. Therefore the triples $(\Sigma, f, \sigma_{2,3,4})$ are equivalent, and represent the same fixed point in the moduli space.

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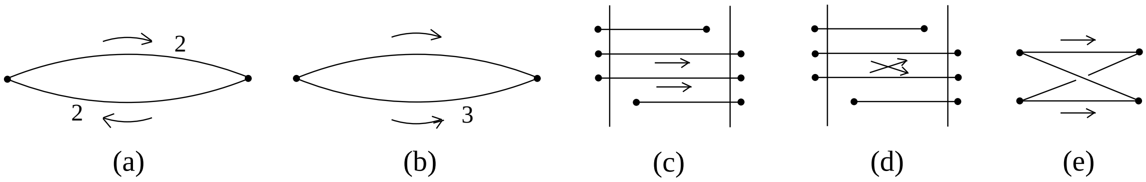

Figure 7: $(\chi, d) = (0, 4)$ symmetric graphs.

This point has an automorphism group of order two. Finally, the contributions of these two graphs must be assigned different signs, as shown in equation (3.37). This assignement is consistent with the sign choices made in the previous example. In fact, we can infer a simple rule for fixing the signs of graphs of this type. The difference between σ_1 and $\sigma_{2,3,4}$ is that σ_1 does not permute the horizontal components, while $\sigma_{2,3,4}$ involve a transposition τ. The sign is simply given by $-$(signature of τ). All sign rules obtained so far can be neatly summarized in the formula $(-1)^{\left[\frac{c-1}{2}\right]}$, where $[r]$ denotes the greatest integer smaller or equal to a given rational number r. We will show below that this rule also works for $(\chi, d) = (0, 4), (1, 5)$, therefore it is tempting to conjecture its validity for arbitrary (χ, d).

To conclude this example, note that the sum of all contributions listed in (3.37) is $C_{1,3} = b_1 = 1/24$ independent of the toric weights. This is again in agreement with large N duality.

vii) $(\chi, d) = (0, 4)$. This case is very similar to $(\chi, d) = (0, 2)$. We have five graphs represented in figure 7.

The individual contributions are

$$\begin{aligned} C_{0,4}^{(a)} &= -\frac{1}{4}\frac{\lambda_u\lambda_v(\lambda_z^2 - 4\lambda_u\lambda_v)}{z^4}, \qquad C_{0,4}^{(b)} = \frac{1}{8}\frac{\lambda_u\lambda_v(2\lambda_z^2 - 9\lambda_u\lambda_v)}{\lambda_z^4}, \\ C_{0,4}^{(c)} &= \frac{1}{2}\frac{\lambda_u^2\lambda_v^2}{\lambda_z^4}, \qquad C_{0,4}^{(d)} = -\frac{1}{2}\frac{\lambda_u^2\lambda_v^2}{\lambda_z^4}, \qquad C_{0,4}^{(e)} = \frac{1}{8}\frac{\lambda_u^2\lambda_v^2}{\lambda_z^4}. \end{aligned} \tag{3.40}$$

Note that for the first graph, the only admissible involution has to permute the two components in order to obtain a symmetric map. We have also used the above set of rules for fixing the signs of graphs three and four. The sum of all individual contributions is zero, as predicted by large N duality. Finally, let us consider our last example

viii) $(\chi, d) = (1, 5)$. This computation is more involved because it involves alltogether fourteen symmetric graphs, which are represented in figure 8.

Since the formulae become quite cumbersome, we will not list all individual contributions here. Let us just note that using the rules specified so far one can obtain the final answer to be $C_{1,5} = b_1$, again in precise agreement with duality predictions. Given the complexity of the cancellations involved in the process, this is a highly nontrivial test of our approach.

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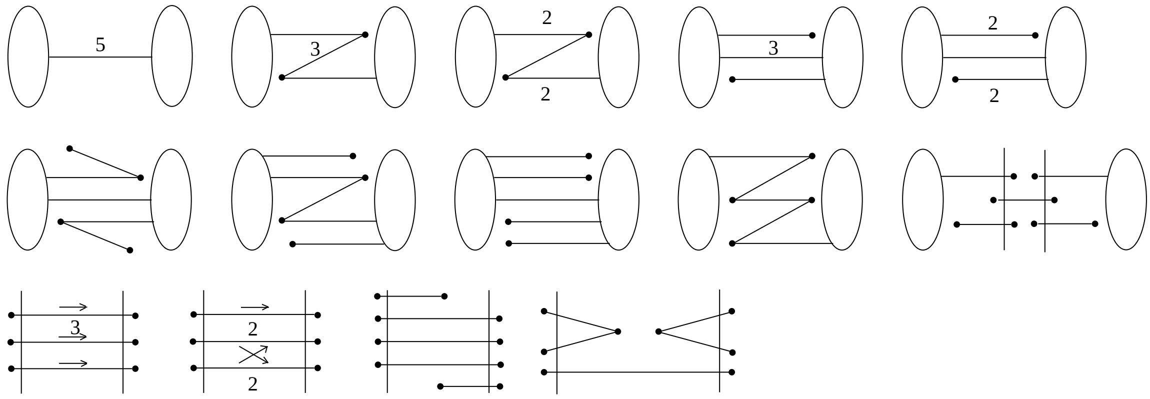

Figure 8: $(\chi, d) = (1,5)$ symmetric graphs.

We conclude this section with a last remark on the free energy of this model. Recall that the duality prediction for this quantity is (3.26). As briefly mentioned earlier, one can reproduce this formula by localization techniques. The main point, as shown in [33, 36] is to make a good choice of weights so that a large number of graphs vanish, and the remaining graphs can be easily evaluated. In our case, we can choose for example $\lambda_u = \lambda_v + \lambda_z = 0$. A consequence of this choice is that only graphs with exactly one edge have nonzero contributions. This phenomenon has been noticed in the context of closed strings in [33, 36] and in the context of open strings in [7, 11]. Then one can easily recover the formula (3.26) using essentially the same manipulations as in the above references. We will not reproduce the details here.

4. Localization and invariant graphs II

So far we have proposed a concrete approach to unoriented closed string enumerative geometry, and applied it in a simple context — the resolved conifold geometry. We claim that this formalism is in fact applicable to any toric Calabi-Yau threefold with antiholomorphic involution. In order to illustrate the method, in this section we will perform localization computations for a more complicated local geometry. The $\mathbb{RP}^2$ results will be succesfully checked against mirror symmetry computations.

The threefold X is now described by the following toric data

$$\begin{array}{ccccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \\ \mathbb{C}^* & 1 & -3 & 1 & 1 & 0 & 0 \\ \mathbb{C}^* & 0 & 1 & -1 & -1 & 1 & 0 \\ \mathbb{C}^* & 0 & 0 & 1 & 1 & -3 & 1. \end{array} \tag{4.1}$$

Note that there are two compact divisors D_1, D_2 on X, each isomorphic to $\mathbb{P}^2$, and a rational $(-1,-1)$ curve C which intersects both divisors transversely. The divisors do not intersect each other.

The antiholomorphic involution is given by

$$I : (X_1, X_2, X_3, X_4, X_5, X_6) \to (\overline{X}_6, \overline{X}_5, \overline{X}_4, -\overline{X}_3, -\overline{X}_2, \overline{X}_1)\,. \tag{4.2}$$

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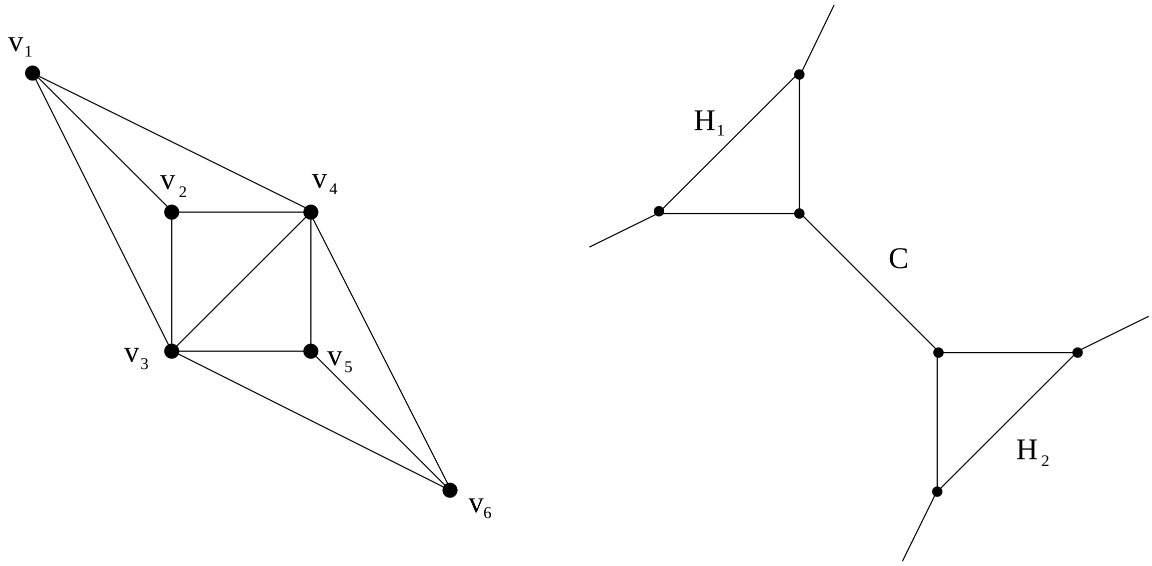

Figure 9: Toric diagram and configuration of invariant curves for the model given in (4.1).

We follow the steps described in the previous section. Consider a real torus action on X of the form

$$e^{i\phi} \cdot (X_1, X_2, X_3, X_4, X_5, X_6) \to (e^{i\lambda_1\phi}X_1, e^{i\lambda_2\phi}X_2, e^{i\lambda_3\phi}X_3, e^{i\lambda_4\phi}X_4, e^{i\lambda_5\phi}X_5, e^{i\lambda_6\phi}X_6)\,. \tag{4.3}$$

This torus action is compatible with the antiholomorphic involution if the following relations between weights are satisfied

$$\lambda_1 + \lambda_6 = 0\,, \qquad \lambda_2 + \lambda_5 = 0\,, \qquad \lambda_3 + \lambda_4 = 0\,. \tag{4.4}$$

The configuration of T-invariant curves on X is represented in figure 9. We have the familiar configurations of invariant hyperplanes in each $\mathbb{P}^2$ connected by the curve C.

The second homology of X is generated by the class of C and the two hyperplane classes of D_1, D_2, which will be denoted by H_1, H_2. Note that I maps D_1 antiholomorphically onto D_2, and it acts as $I_*(C) = -C$, $I_*(H_1) = -H_2$ on homology. Therefore the second homology of the quotient space X/I is generated by C/I, which is an $\mathbb{RP}^2$ cycle as before, and a second class H. The Kähler class ω is compatible with the action on homology if $I^*\omega = -\omega$.

The closed string Gromov-Witten invariants of X can be computed by localization as explained before. The algorithm is very similar to the previous case, except that the structure of the fixed loci is more complicated. In fact one can extend without too much effort the graph representation to the present case [24]. The same labeling rules apply except that the integer k_v can now take values $k_v = 1, \ldots, 6$ corresponding to the six fixed points $P_1, \ldots, P_6$ of the torus action on X. The edges $e \in E(\Gamma)$ represent irreducible rational components of Σ which are mapped $d_e : 1$ to one of the invariant curves on X. For convenience, an edge e will be called of type (ij), with $i, j = 1, \ldots, 6$ is the image $f(\Sigma_e)$ in X is the invariant curve $\overline{P_iP_j}$. We will also denote by $E_{(ij)}(\Gamma)$ the set of edges of Γ of given

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type (ij). The local contribution of an arbitrary graph can be worked as in the previous section using normalization exact sequences. We will not attempt to write down an explicit formula since it would be too complicated. Concrete computations will be performed in several examples below.

In the unoriented sector of the theory, we have to sum over triples (Σ, f, σ) as in above. This means that the corresponding fixed loci can be classified by symmetric graphs. A symmetric graph is defined to be a Kontsevich graph equipped with two involutions $\tau_E : E(\Gamma) \to E(\Gamma)$, $\tau_V : V(\Gamma) \to V(\Gamma)$ subject to certain conditions analogous to $(I) - -(III)$ in the previous section. Let us make this more explicit.

(I') The action of τ_V on vertices should be compatible with the target space involution I, $k_{\tau_V(v)} = k_v + 3$, and it should leave the genus invariant, $g_{\tau_V(v)} = g_v$.

(II') The action of τ_E on edges should be compatible with target space involution. This means that there should be a precise correlation between the type of e and the type of $\tau_E(e)$ as follows

$$\begin{aligned} e \in E_{(14)}(\Gamma) &\Rightarrow \tau_E(e) \in E_{(14)}(\Gamma) \\ e \in E_{(12)}(\Gamma) &\Rightarrow \tau_E(e) \in E_{(45)}(\Gamma) \\ e \in E_{(13)}(\Gamma) &\Rightarrow \tau_E(e) \in E_{(46)}(\Gamma) \\ e \in E_{(23)}(\Gamma) &\Rightarrow \tau_E(e) \in E_{(56)}(\Gamma)\,. \end{aligned} \tag{4.5}$$

Moreover, τ_E should leave the degree invariant $d_{\tau_E(e)} = d_e$ for any $e \in E(\Gamma)$; if $\tau_E(e) = e$, d_e should be odd.

(III') The involutions τ_V, τ_E must be compatible i.e. if (v, e) is a flag, then $(\tau_V(v), \tau_E(e))$ must be also a flag.

Given these conditions, it is fairly straightforward to work out the local contribution of a symmetric graph. Essentially, the new element compared to the previous section is the presence of extra vertices and edges mapping to D_1, D_2, but not C. A vertex or edge mapping to D_1 is always pairwise identified to a vertex or edge mapping to D_2. For each such pair of vertices $(v, \tau_V(v))$ we will include a single integral $\int_{\overline{M}_{g_v,val(v)}}$ by analogy with the previous case. Similarly for each pair of edges $(e, \tau_E(e)) \in (E(\Gamma) \setminus E_{(14)}(\Gamma))^2$ we include only one edge factor. If the torus action is compatible with the involution I, the edge factors of e and $\tau_E(e)$ are equal, hence there is no ambiguity here. If $e \in E_{(14)}(\Gamma)$, then $f(\Sigma_e) = C$, and we apply the rules of the previous section. Therefore the only ambiguities present in this approach are the ones encountered in the previous computation for which have a simple set of rules. We will show in the following that the same set of rules applies to the present case without modification. This is a nontrivial consistency check of the methods developed here.

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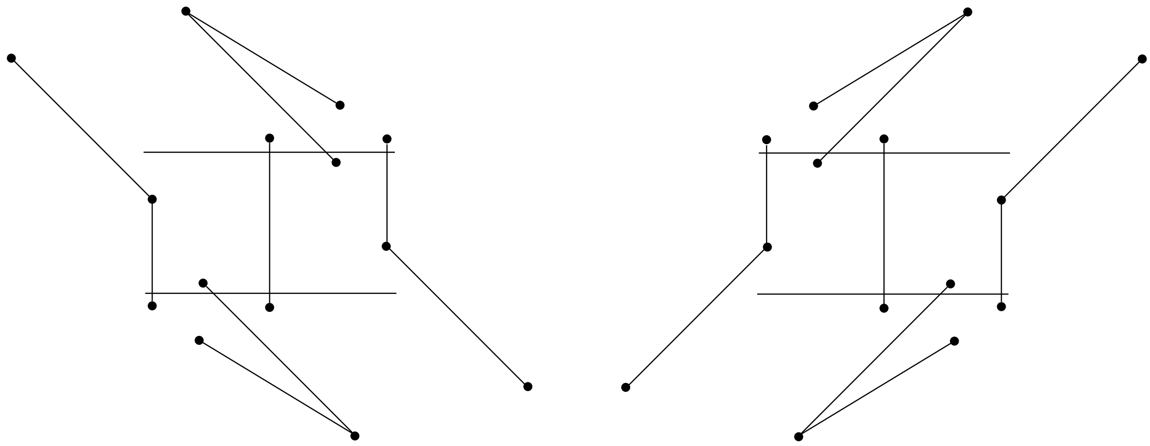

Figure 10: A pair of $(\chi, m, n) = (-1, 3, 3)$ symmetric graphs.

4.1 Results

Let us now present some results obtained using this algorithm. The unoriented free energy has an expansion of the form

$$\mathcal{F}_{(X,I)} = \sum_{h\geq 0}\sum_{c\geq 1} g_s^{-\chi} \sum_{\substack{m,n\geq 0 \\ (m,n)\neq 0}} C_{\chi,m,n} q_1^n q_2^{m/2} \,, \tag{4.6}$$

where (m, n) are defined by $f_*[\Sigma] = mC + n(H_1 + H_2)$. Note that any symmetric map must have equal degrees with respect to H_1, H_2. The coefficients $C_{\chi,m,n}$ can be computed by summing over all symmetric graphs compatible with the triple (χ, m, n). This is a fairly straightforward, although tedious process. If we carefully employ all the rules found in the previous section, we find the following expansion

$$\begin{aligned}\mathcal{F}_{(X,I)} = \frac{1}{g_s}\Big(& q_2^{1/2} - 2q_1 q_2^{1/2} + 5q_1^2 q_2^{1/2} - 32 q_1^3 q_2^{1/2} + \cdots + \frac{1}{9} q_2^{3/2} - 3q_1^2 q_2^{3/2} + \\ & + \frac{268}{9} q_1^3 q_2^{3/2} + \cdots \Big) + g_s \left(\frac{1}{24} q_2^{1/2} - \frac{1}{12} q_1 q_2^{1/2} + \cdots \right) .\end{aligned} \tag{4.7}$$

A couple of remarks are in order here. Note that all coefficients $C_{\chi,m,n}$ are independent of weights and satisfy the expected integrality properties. This is a nontrivial result, given the fact that the number of graphs increases very rapidly, and the local contributions become quite complicated. For example the coefficient $C_{-1,3,3} = 268/9$, which has the correct multicover behavior, is the result of summing one hundred and seventy-eight symmetric graphs, which can be naturally grouped in pairs. A typical pair of graphs that occurs in this computation is represented below.

The local contribution of this pair is of the form

$$\frac{1}{2} \frac{-3\lambda_u^5 \lambda_v + 2\lambda_u^4 \lambda_v^2 + \lambda_u^6 - 3\lambda_u \lambda_v^5 + 2\lambda_u^2 \lambda_v^4 + \lambda_v^6}{\lambda_z^2 \lambda_u^2 \lambda_v^2} \,. \tag{4.8}$$

Next, several coefficients on this expansion turn out to be zero, as a result of nontrivial cancellations among different graphs. For example we have $C_{-1,3,1} = C_{-1,5,1} = 0$. At higher

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genus, $\chi = 1$ we obtain the expected multicover formulae, that is $C_{1,1,1} = -\frac{1}{12} = C_{-1,1,1}b_1$, and $C_{1,3,1} = 0 = C_{-1,3,1}b_1$. In the last case one has to sum over fifteen pairs of graphs, many of them involving integrals over $\overline{M}_{1,k}$ with various k. Moreover, at this level, we have also to use the fix signs as discussed in the previous section. In principle, one could compute higher order terms in the expansion, but the computations become very tedious. We will not further pursue this aspect here. Below we compare the above results with $\chi = -1$ to mirror symmetry computations.

4.2 Mirror symmetry

In this section we perform a **B**-model computation for the $\mathbb{RP}^2$ free energy of the above model. We will find that the **B**-model expansion is in precise agreement with the **A**-model computation (4.7) up to an overall factor of 2 which has been explained in section 2 (second paragraph below equation (2.4).)

The local mirror geometry of X is a Landau-Ginzburg model described in terms of six dual variables $y_1, \ldots, y_6 \in (\mathbb{C}^*)^6$ subject to the following relations [37, 38]

$$\begin{aligned} y_1y_3y_4 &= e^{-t_1}y_2^3\,, \\ y_2y_5 &= e^{-t_2}y_3y_4\,, \\ y_3y_4y_6 &= e^{-t_3}y_5^3\,. \end{aligned} \tag{4.9}$$

Here t_1, t_2, t_3 are complexified Kähler parameters, and the relations follow from the generators of the Mori cone of X. The Landau-Ginzburg superpotential is

$$W_{LG} = \sum_{i=1}^{6} y_i\,. \tag{4.10}$$

We work in the patch $y_4 = 1$ and set $y_2 = e^v, y_3 = e^u$, with $u, v \sim u + 2\pi, v + 2\pi$. Using the relations (4.9), we can now rewrite the local mirror equation (4.10) as

$$e^{-t_1+3v-u} + e^{-t_2+u-v} + e^{-3t_2-3t_3+2u-3v} + e^u + e^v + 1 = 0\,. \tag{4.11}$$

This Riemann surface determines a three dimensional hypersurface

$$xz = e^{-t_1+3v-u} + e^{-t_2+u-v} + e^{-3t_2-3t_3+2u-3v} + e^u + e^v + 1 \tag{4.12}$$

in $(\mathbb{C}^*)^2 \times \mathbb{C}^2$, which is the local mirror of X.

Following the general considerations of [15], the antiholomorphic involution $I : X \to X$ should be mapped by mirror symmetry to a holomorphic involution J of the hypersurface (4.12). Moreover, the $\mathbb{RP}^2$ partition function (or superpotential) in the **B**-model is related to a one-chain period of the holomorhic one-form udv on the Riemann surface (4.11). The boundary of the one-chain is specified by the fixed loci of the holomorphic involution J. In particular, if J acts freely, the $\mathbb{RP}^2$ free energy is zero. The relation between the superpotential and the chain period is straightforward for the resolved conifold geometry, but somewhat subtle in the present case. We will discuss this in detail below.

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Using the methods of [15], we find that the hypersurface (4.12) admits a mirror involution acting as

$$(x, z, e^v, e^u) \to (-x, -ze^{-u}, e^{-t_2-v}, e^{-u}) . \tag{4.13}$$

if the complex structure moduli satisfy $z_1 = z_3$. This condition is related by mirror symmetry to the condition $t_1 = t_3$ on the **A**-model side, which follows from the action of I on the Kähler class. The fixed point set has two components determined by $v = t_2/2$ and respectively $v = t_2/2 + i\pi$.

In principle, the superpotential should be computed by integrating the holomorphic one-form udv on a one-chain on the Riemann surface stretching between these two points. This is very similar to the computation of the superpotential for holomorphic branes [5, 8, 12]. Here we will mainly follow the approach of [8, 12]. In that case one has an algebraic coordinate z_0 on the D-brane moduli space which can be in this case identified to v. The superpotential is a function $W(z_0, z_1, z_2, z_3)$ which has been shown in [8, 12] to be a double logarithmic solution of a GKZ system. In order to obtain an expansion with correct integrality properties, we have to expand this function in terms of flat coordinates q_0, q_1, q_2, q_3. q_0 is an open string flat coordinate associated to z_0, and $q_{1,2,3}$ are standard closed string flat coordinates.

Our problem is mathematically very similar to the open string computation. At the classical level, we can formally think of the two orientifold planes as two holomorphic branes located at the points $z_0 = \pm z_2^{1/2}$. We stress that this is just a formal analogy, motivated by the mathematical similarities between the two systems. We do not consider open string theories in this paper. However, this is the correct interpretation only at classical level. At quantum level, the correct coordinate on the D-brane moduli is the flat coordinate q_0. Therefore the positions of the two orientifold planes should be corrected to $q_0 = \pm q_2^{1/2}$. Moreover, the condition $z_1 = z_3$ translates into $q_1 = q_3$. In conclusion, we claim that the exact $\mathbb{RP}^2$ superpotential should be given by the following expression

$$W_{\mathbb{RP}^2} = W(q_2^{1/2}, q, q_2, q) - W(-q_2^{1/2}, q, q_2, q) , \tag{4.14}$$

where $q = q_1 = q_3$. Using the results of [8, 12] we find

$$W = \sum_{n_0,n_1,n_2,n_3} \frac{z_0^{n_0}(-z_1)^{n_1}(-z_2)^{n_2}(-z_3)^{n_3}\Gamma(n_0)\Gamma(n_0 - n_1 + n_2 - n_3)}{\Gamma(1+n_0)\Gamma(1+n_1)\Gamma(1+n_3)\Gamma(1+n_0-3n_1+n_2)\Gamma(1+n_1-n_2+n_3)} \times \\ \times \frac{1}{\Gamma(1+n_2-3n_3)} . \tag{4.15}$$

The open and closed flat coordinates are given by

$$q_0 = z_0 e^{-f_1} , \qquad q_1 = z_1 e^{3f_1} , \qquad q_2 = z_2 e^{-f_1-f_3} , \qquad q_3 = z_3 e^{3f_3} , \tag{4.16}$$

where

$$\begin{aligned} f_1 &= \sum_{n_1,n_2,n_3} \frac{(-z_1)^{n_1}(-z_2)^{n_2} z_3^{n_3}\Gamma(3n_1 - n_2)}{\Gamma(1+n_1)\Gamma(1+n_3)\Gamma(1+n_1-n_2+n_3)^2\Gamma(1+n_2-3n_3)} , \\ f_3 &= \sum_{n_1,n_2,n_3} \frac{z_1^{n_1}(-z_2)^{n_2}(-z_3)^{n_3}\Gamma(-n_2+3n_3)}{\Gamma(1+n_1)\Gamma(1+n_3)\Gamma(1+n_1-n_2+n_3)^2\Gamma(1-3n_1+n_2)} . \end{aligned} \tag{4.17}$$

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In terms of flat coordinates, the superpotential has the following expansion

$$\begin{aligned}W =& q_0 + \frac{1}{4}q_0^2 - 2q_0q_1 + \frac{1}{9}q_0^3 - q_0^2q_1 + 5q_0q_1^2 + q_0q_1q_2 + \frac{1}{16}q_0^4 - q_0^3q_1 + \frac{7}{2}q_0^2q_1^2 + q_0^2q_1q_2 - \\ &- 32q_0q_1^3 - 4q_0q_1^2q_2 - 2q_0q_1q_2q_3 + \frac{1}{25}q_0^5 - q_0^4q_1 + 3q_0^3q_1^2 + q_0^3q_1q_2 - 21q_0^2q_1^3 - \\ &- 3q_0^2q_1^2q_2 - 2q_0^2q_1q_2q_3 - 5489q_0q_1^4 + 35q_0q_1^3q_2 + 8q_0q_1^2q_2q_3 + q_0q_1q_2^2q_3 + \\ &+5q_0q_1q_2q_3^2 + \cdots - q_0^5q_1 - \frac{164}{9}q_0^3q_1^3 + \cdots .\end{aligned} \tag{4.18}$$

Therefore we obtain

$$\begin{aligned}W =& 2q_2^{1/2} - 4q_1q_2^{1/2} + 10q_1^2q_2^{1/2} - 64q_1^3q_2^{1/2} + \cdots + \frac{2}{9}q_2^{3/2} - 6q_1^2q_2^{3/2} + \frac{536}{9}q_1^3q_2^{3/2} + \cdots + \\ &+ \frac{2}{25}q_2^{5/2} + \cdots .\end{aligned} \tag{4.19}$$

This expansion is in precise agreement with the $\mathbb{RP}^2$ free energy computed in (4.7).

Acknowledgments

We would like to thank Bobby Acharya and Harald Skarke for useful discussions and correspondence.

A. Edge factors for symmetric maps

In this appendix we compute the edge factors for horizontal components Σ_e which are preserved by the antiholomorphic involution σ, that is $\tau_E(e) = e$. In this case d_e must be odd, and $f_e : \Sigma_e \to X$ is a $d_e : 1$ cover of the $(-1,-1)$ curve $C \subset X$. To simplify notation, throughout this section, we will drop the subscript e from Σ_e, f_e, d_e. In terms of local coordinates, f is given by $z = w^d$. Note that there is a induced torus action on the domain Σ_e with weight $\lambda_w = \lambda_z/d$. The edge factor of this map is

$$C(e) = \frac{e_T\left(H^1(f^*(T_X))^{(\sigma,I),m}\right) e_T\left(H^0(\Sigma,T_\Sigma)^{\sigma,m}\right)}{e_T\left(H^0(f^*(T_X))^{(\sigma,I),m}\right)}. \tag{A.1}$$

The superscript (σ, I) or σ indicates that the we have to take the invariant part of the cohomology groups under the induced involution, as discussed in section three. The superscript m denotes the moving part with respect to the torus action, as usual.

A straightforward computation shows that the cohomology groups are given by

$$\begin{aligned}H^0(\Sigma,T_\Sigma) &= \left\{(a_0 + a_1w + a_2w^2)\partial_w\right\} \\ H^0(\Sigma, f^*T_X) &= \left\{(b_0 + b_1w + \cdots + b_{2d}w^{2d})\partial_z\right\} \\ H^1(\Sigma, f^*T_X)^\vee &= \left\{(c_0 + c_1w + \cdots + c_{d-2})dw \otimes \partial_u + (d_0 + d_1w + \cdots + d_{d-2})dw \otimes \partial_v\right\}.\end{aligned} \tag{A.2}$$

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In the last line we have used Kodaira-Serre duality, $H^1(\Sigma, f^*T_X)^\vee \simeq H^0(\Sigma, f^*(T_X^\vee) \otimes \mathcal{O}(K_\Sigma))$. The induced antiholomorphic involutions act as

$$\begin{aligned}
\sigma: \quad & a_n \to (-1)^n \overline{a}_{2-n}, && n = 0, 1, 2 \\
(\sigma, I): \quad & b_n \to (-1)^n \overline{b}_{2d-n}, && n = 0, \ldots, 2d \\
(\sigma, I): \quad & c_n \to (-1)^{d-2-n} \overline{d}_{d-2-n}, && n = 0, \ldots, d-2 \\
(\sigma, I): \quad & d_n \to -(-1)^{d-2-n} \overline{c}_{d-2-n}, && n = 0, \ldots, d-2\,.
\end{aligned} \tag{A.3}$$

Therefore the fixed subspaces are characterized by

$$\begin{aligned}
H^0(\Sigma, T_\Sigma)^\sigma: \quad & a_0 = \overline{a}_2,\ a_1 = -\overline{a}_1 \\
H^0(\Sigma, f^*T_X)^{(\sigma,I)}: \quad & b_n = (-1)^n \overline{b}_{2d-n}, && n = 0, \ldots, 2d \\
\left(H^1(\Sigma, f^*T_X)^\vee\right)^{(\sigma,I)}: \quad & c_n = (-1)^{d-2-n} \overline{d}_{d-2-n}, && n = 0, \ldots, d-2.
\end{aligned} \tag{A.4}$$

Note that these real subspaces are not equipped with a canonical complex structure, but they are equipped with a canonical complex structure up to conjugation. Therefore the edge factor (A.1) has an ambiguity of order two which can be fixed by choosing a complex structure on the fixed subspaces. We do not know how these choices must be correlated from the first principles. Here we will make some arbitrary choices, and fix the signs during the computation as explained in section 3.1. Then we obtain

$$\begin{aligned}
H^0(\Sigma, T_\Sigma)^{\sigma,m} &\simeq \left(-\frac{\lambda_z}{d}\right) \\
H^0(\Sigma, f^*T_X)^{(\sigma,I),m} &\simeq (-\lambda_z) \oplus \left(-\frac{(d-1)\lambda_z}{d}\right) \oplus \ldots \oplus \left(-\frac{\lambda_z}{d}\right) \\
H^1(\Sigma, f^*T_X)^{(\sigma,I),m} &\simeq \left(-\lambda_v - \frac{\lambda_z}{d}\right) \oplus \left(-\lambda_v - \frac{2\lambda_z}{d}\right) \oplus \ldots \left(-\lambda_v - \frac{(d-1)\lambda_z}{d}\right).
\end{aligned} \tag{A.5}$$

By substituting this back in equation (A.1), we obtain formula (3.25) in the main text.

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